Curl

Defn: The curl of vector field $\vec{v} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ is

$$\text{curl}(\vec{v}) = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle.$$

NB: It is easier to remember the symbolic formula

$$* \ \text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}.$$

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z} \right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle$$

Ex: Compute $\text{curl}(\vec{v})$ for $\vec{v} = \langle xy, xyz, -y^2 \rangle$.

Sol:
$$\text{curl}(\vec{v}) = \nabla \times \vec{v}$$

$$= \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xyz & -y^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-y^2] - \frac{\partial}{\partial z}[xyz], -\left(\frac{\partial}{\partial x}[-y^2] - \frac{\partial}{\partial z}[xy] \right), \frac{\partial}{\partial x}[xyz] - \frac{\partial}{\partial y}[xy] \right\rangle$$

$$= \langle -2y - xy, -(0-0), yz - x \rangle = \langle -y(x+2), 0, yz - x \rangle \ \boxed{//}$$

Prop: A vector field $\vec{v}$ on $\mathbb{R}^3$ with continuous partial derivatives is conservative precisely when $\operatorname{curl}(\vec{v}) = \vec{0}$.

pf ($\Rightarrow$): We show that conservative implies curl $= \vec{0}$.

Assume $\vec{v}$ is conservative. Hence $\vec{v} = \nabla f$ for some $f: \mathbb{R}^3 \to \mathbb{R}$. Because $\vec{v}$ has continuous partial derivatives, f satisfies Clairaut's Theorem. Hence (for $\vec{v} = \langle P, Q, R \rangle$):

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} = \frac{\partial Q}{\partial x}, \text{ so } \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0. \ast$$

$$\frac{\partial R}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{\partial P}{\partial z}, \text{ so } -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = 0. \ast$$

$$\frac{\partial R}{\partial y} = \frac{\partial}{\partial y}\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\frac{\partial f}{\partial y} = \frac{\partial Q}{\partial z}, \text{ so } \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} = 0. \ast$$

On the other hand, we compute the curl of $\vec{v}$ by:

$$\operatorname{curl}(\vec{v}) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial z} \right\rangle$$

$$= \langle 0, 0, 0 \rangle = \vec{0}. \qquad \boxed{///}$$

Ex: Is $\vec{v} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle$ Conservative? If so, compute a potential function for $\vec{v}$.

Sol: Curl $(\vec{v}) = \vec{0}$ if and only if $\vec{v}$ is conservative.

$$\text{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}\left[3xy^2z^2\right] - \frac{\partial}{\partial z}\left[2xyz^3\right], -\left(\frac{\partial}{\partial x}\left[3xy^2z^2\right] - \frac{\partial}{\partial z}\left[y^2z^3\right]\right), \frac{\partial}{\partial x}\left[2xyz^3\right] - \frac{\partial}{\partial y}\left[y^2z^3\right] \right\rangle$$

$$= \langle 6xyz^2 - 6xyz^2, -(3y^2z^2 - 3y^2z^2), 2yz^3 - 2yz^3 \rangle = \langle 0, 0, 0 \rangle$$

Hence $\vec{v} = \langle y^2z^3, 2xyz^3, 3xy^2z^2 \rangle = \nabla f$ is conservative.

$\frac{\partial f}{\partial x} = y^2z^3$, so $f(x,y,z) = \int y^2z^3\,dx = xy^2z^3 + C(y,z)$

$2xyz^3 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[xy^2z^3 + C(y,z)\right] = 2xyz^3 + C_y(y,z)$

$C_y(y,z) = 0$ so $C(y,z) = \int 0\,dy = D(z)$

$3xy^2z^2 = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[xy^2z^3 + D(z)\right] = 3xy^2z^2 + D'(z)$

$\therefore D'(z) = 0$, so $D(z) = \int 0\,dz = E$.

Hence $f(x,y,z) = xy^2z^3 + E$ for some E. //

Ex: Is $\vec{V} = \langle xyz^4, x^2z^4, 4x^2yz^3 \rangle$ Conservative?

If so, compute a potential function for $\vec{V}$.

Sol: $\text{Curl}(\vec{v}) = \vec{0}$ if and only if $\vec{v}$ is conservative.

$$\text{Curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz^4 & x^2z^4 & 4x^2z^3 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[4x^2z^3] - \frac{\partial}{\partial z}[x^2z^4], -\left(\frac{\partial}{\partial x}[4x^2z^3] - \frac{\partial}{\partial z}[xyz^4]\right), \frac{\partial}{\partial x}[x^2z^4] - \frac{\partial}{\partial y}[xyz^4] \right\rangle$$

$$= \langle 0 - 4x^2z^3, -(8xz^3 - 4xyz^3), 2xz^4 - xz^4 \rangle$$

$$= \langle -4x^2z^3, -4xz^3(2-y), xz^4 \rangle = xz^3\langle -4x, 4(y-2), z \rangle \neq \vec{0}$$

$\therefore$ $\vec{V}$ is not conservative! ☑

Ex: Is vector field $\vec{v}$ Conservative?

If so, compute a potential function for $\vec{v}$.

① $\vec{V} = \langle z\cos(y), xz\sin(y), x\cos(y) \rangle$

② $\vec{V} = \langle 1, \sin(z), y\cos(z) \rangle$

③ $\vec{V} = \langle e^x\sin(yz), ze^x\cos(yz), ye^x\cos(yz) \rangle$

④ $\vec{V} = \langle e^{yz}, xze^{yz}, xye^{yz} \rangle$

Sol: Left to you as an exercise :) ☑

Divergence

Def^n: The <u>divergence</u> of vector field $\vec{v} = \langle P, Q, R \rangle$ on $\mathbb{R}^3$ is

$$\operatorname{div}(\vec{v}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$

NB: It is easier to remember the symbolic formula

$$\operatorname{div}(\vec{v}) = \nabla \cdot \vec{v} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \vec{v}.$$

Ex: Compute the divergence of $\vec{v} = \langle xz, xyz, -y^2 \rangle$.

Sol: $\operatorname{div}(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial}{\partial x}[xz] + \frac{\partial}{\partial y}[xyz] + \frac{\partial}{\partial z}[-y^2] = z + xz$ ☐

Ex: Compute the divergence of $\vec{v} = \langle xe^{yz}, \cos(xy), -\sin(xz) \rangle$

Sol: $\operatorname{div}(\vec{v}) = \nabla \cdot \vec{v} = \frac{\partial}{\partial x}[xe^{yz}] + \frac{\partial}{\partial y}[\cos(xy)] + \frac{\partial}{\partial z}[-\sin(xz)]$

$= e^{yz} - x\sin(xy) - x\cos(xz)$ ☐

Ex: Compute $\operatorname{div}(\operatorname{curl}(\vec{v}))$ for $\vec{v} = \langle xz, xyz, -y^2 \rangle$.

Sol:

$$\operatorname{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{bmatrix}$$

$$= \left\langle \frac{\partial}{\partial y}[-y^2] - \frac{\partial}{\partial z}[xyz], -\left(\frac{\partial}{\partial x}[-y^2] - \frac{\partial}{\partial z}[xz]\right), \frac{\partial}{\partial x}[xyz] - \frac{\partial}{\partial y}[xz] \right\rangle$$

$$= \langle -2y - xy, -(0 - x), yz - 0 \rangle = \langle -y(x+2), x, yz \rangle$$

$$\operatorname{div}(\operatorname{curl}(\vec{v})) = \frac{\partial}{\partial x}[-y(x+z)] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[yz]$$

$$= -y + 0 + y = 0$$ ☐

Prop: If $\vec{v}$ is a vector field on $\mathbb{R}^3$ with continuous second-order partial derivatives, then $\operatorname{div}(\operatorname{curl}(\vec{v})) = 0$.

pf: Let $\vec{v} = \langle P, Q, R \rangle$. Then

$$\operatorname{curl}(\vec{v}) = \nabla \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{bmatrix}$$

$$= \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, -\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right), \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle, \quad \text{so}$$

$$\operatorname{div}(\operatorname{curl}(\vec{v})) = \nabla \cdot \operatorname{curl}(\vec{v})$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] + \frac{\partial}{\partial y}\left[-\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]$$

$$= \boxed{\frac{\partial^2 R}{\partial x \partial y}} - \boxed{\frac{\partial^2 Q}{\partial x \partial z}} - \boxed{\frac{\partial^2 R}{\partial y \partial x}} + \boxed{\frac{\partial^2 P}{\partial y \partial z}} + \boxed{\frac{\partial^2 Q}{\partial z \partial x}} - \boxed{\frac{\partial^2 P}{\partial z \partial y}}$$

$$= \left(\frac{\partial^2 P}{\partial y \partial z} - \frac{\partial^2 P}{\partial z \partial y}\right) + \left(\frac{\partial^2 Q}{\partial z \partial x} - \frac{\partial^2 Q}{\partial x \partial z}\right) + \left(\frac{\partial^2 R}{\partial x \partial y} - \frac{\partial^2 R}{\partial y \partial x}\right)$$

$$= 0 + 0 + 0 = 0$$ by Clairaut's Theorem ☐

Ex: Is $\underline{\vec{v} = \langle x, \cos(xy), \sin(yz) \rangle}$ the curl of a vector field?

Sol: If $\vec{v} = \operatorname{curl}(\vec{u})$, then $\operatorname{div}(\vec{v}) = \operatorname{div}(\operatorname{curl}(\vec{u})) = 0 \ldots$

$$\operatorname{div}(\vec{v}) = \frac{\partial}{\partial x}[x] + \frac{\partial}{\partial y}[\cos(xy)] + \frac{\partial}{\partial z}[\sin(yz)]$$

$$= 1 - x\sin(xy) + y\cos(yz) \neq 0$$

$\therefore$ $\vec{v}$ is <u>not</u> the curl of any vector field. ☐

Properties of Curl and Divergence

Let's make sure we understand the inner-workings of grad, Curl, and divergence for vector and scalar fields

Ex: Let $\vec{v}$ be a vector field on $\mathbb{R}^3$ and f a scalar field.

Which of the following expressions make sense?

① $\text{curl}(f)$ — Nonsense...

② ∇f — Gradient :)

③ $\text{div}(f)$ — Nonsense!

④ $\text{curl}(\vec{v})$ — :)

⑤ $\nabla \vec{v}$

$\nabla \vec{v} = \left\langle \frac{\partial v_x}{\partial x}, \frac{\partial v_y}{\partial y}, \frac{\partial v_z}{\partial z} \right\rangle$

$\nabla^2 \vec{v} = \left\langle \frac{\partial^2 v_x}{\partial x^2}, \frac{\partial^2 v_y}{\partial y^2}, \frac{\partial^2 v_z}{\partial z^2} \right\rangle$

⑥ $\text{div}(\vec{v})$ — :)

⑦ $\text{Curl}(\text{curl}(\vec{v}))$, where $\text{curl}(\vec{v})$ is a v.f. — :)

⑧ $\nabla(\text{curl}(\vec{v}))$, where $\text{curl}(\vec{v})$ is a v.f. — :)

⑨ $\text{div}(\text{curl}(\vec{v})) = 0$, where $\text{curl}(\vec{v})$ is a v.f. — :)

⑩ $\nabla(\nabla f)$, where ∇f is a v.f. — :)

⑪ $\text{div}(\nabla f)$, where ∇f is a v.f. — :)

⑫ $\text{div}(\text{div}(\vec{v}))$, where $\text{div}(\vec{v})$ is a s.f. — Nonsense.

⑬ $\nabla(\text{div}(\vec{v}))$, where $\text{div}(\vec{v})$ is a s.f. — :)

⑭ $\text{curl}(\nabla f) = \vec{0}$, where ∇f is a v.f. — :)

⑮ $\text{curl}(\text{div}(\vec{v}))$, where $\text{div}(\vec{v})$ is a s.f. — Nonsense!

Prop: Suppose f is a scalar field and $\vec{u}, \vec{v}$ are vector fields on $\mathbb{R}^3$. The following relationships all hold.

1. $\operatorname{div}(\vec{u}+\vec{v}) = \operatorname{div}(\vec{u}) + \operatorname{div}(\vec{v})$. ← "Sum laws"
2. $\operatorname{curl}(\vec{u}+\vec{v}) = \operatorname{curl}(\vec{u}) + \operatorname{curl}(\vec{v})$. ←
3. $\operatorname{div}(f\vec{v}) = f\operatorname{div}(\vec{v}) + \nabla f \cdot \vec{v}$ ← "product rule"
4. $\operatorname{div}(\vec{u}\times\vec{v}) = \operatorname{curl}(\vec{u})\cdot\vec{v} - \vec{u}\cdot\operatorname{curl}(\vec{v})$ ←
5. $\operatorname{curl}(\operatorname{curl}(\vec{v})) = \nabla(\operatorname{div}(\vec{v})) - \nabla^2\vec{v}$ ← oddball.

Pf: Left as an exercise

GREEN'S IDENTITIES AND HARMONIC FUNCTIONS

First we'll restate Green's Theorem with curl and divergence.

Prop (GREEN'S THEOREM): Let $\vec{v} = \langle P(x,y), Q(x,y), 0 \rangle$ have continuous partial derivatives on an open region containing closed region D. If ∂D is a piecewise-smooth closed curve, then

① $\iint_D \text{curl}(\vec{v}) \cdot \vec{k} \, dA = \int_{\partial D} \vec{v} \cdot d\vec{r}$ and

② $\int_{\partial D} \vec{v} \cdot \vec{n} \, ds = \iint_D \text{div}(\vec{v}) \, dA$ where

$\vec{n}$ is the outward unit normal vector of a parametrization of ∂D.

pf:

Prop (Green's Identities): Let D be a closed region with piecewise-smooth boundary. If f and g have continuous second-order partial derivatives on an open region containing D, then the following identities hold where $\vec{n}$ is the outward unit normal of a parametrization of ∂D:

① $$\iint_D f \nabla^2 g \, dA = \int_{\partial D} f \nabla g \cdot \vec{n} \, ds - \iint_D \nabla f \cdot \nabla g \, dA$$

② $$\iint_D (f \nabla^2 g - g \nabla^2 f) \, dA = \int_{\partial D} (f \nabla g - g \nabla f) \cdot \vec{n} \, ds$$

pf:

<u>Defⁿ</u>: Function f is <u>harmonic</u> on $D \subseteq \mathbb{R}^n$ when $\nabla^2 f = 0$ on D.

<u>Prop</u>: Assume f is harmonic on $D \subseteq \mathbb{R}^2$ and f has continuous second-order partial derivatives. If ∂D is piecewise-smooth and $\vec{n}$ is the outward unit normal of a parametrization of ∂D, then the following both hold:

① $\displaystyle\int_{\partial D} D_{\vec{n}} f \, ds = 0$

② If $f(x,y) = 0$ on ∂D, then $\displaystyle\iint_D |\nabla f|^2 \, dA = 0$.

<u>pf</u>: